M337/Practice exam 3

Module Examination

Complex analysis

Time allowed: 3 hours

There are **two parts** to this examination.

In **Part 1** you should **submit answers to all 6 questions**. Each question is worth 10% of the total mark.

In **Part 2** you should **submit answers to 2 out of the 3 questions**. Each question is worth 20% of the total mark.

Do not submit more than two answers for Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Include all your working, as some marks are awarded for this.

Write your answers in **pen**, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Follow the instructions in the online timed examination for how to submit your work. Further information about completing and submitting your examination work is in the *Instructions and guidance for your remote examination* document on the module website.

Part 1

You should **submit answers to all questions** from Part 1.

Each question is worth 10%.

Question 1

(a) Express each of the following complex numbers in *polar* form, simplifying your answers as far as possible.

(i) $(-1+i\sqrt{3})^4$ *[3]*

(ii) $(-1)^{3i}$ *[3]*

(b) Find the principal fourth root of

$$w = \frac{1}{1+i}$$

in *polar* form. *[4]*

Question 2

Locate the singularities of each of the following functions, and classify each singularity as a removable singularity, a pole (stating its order) or an essential singularity.

(a) $f(z) = \dfrac{\sin z}{z-\pi}$ *[3]*

(b) $f(z) = \dfrac{e^z - 1}{z^2}$ *[4]*

(c) $f(z) = \cosh \dfrac{1}{z}$ *[3]*

Question 3

(a) Find the residues of the function

$$f(z) = \frac{1}{z^3 - 1}$$

at its poles. *[4]*

(b) Use part (a) to evaluate the integral

$$\int_\Gamma \frac{1}{z^3-1}\,dz,$$

where Γ is the circle $\{z : |z-1| = 1\}$ *[2]*

(c) Use part (a) to evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^3-1}\,dt.$$ *[4]*

Question 4

Let $z = x + iy$.

(a) Prove that $|e^z| = e^x$. *[2]*

(b) Use the Triangle Inequality to prove that

$$|\sinh z| \leq \cosh x.$$ *[3]*

(c) Use part (b) to deduce that the series

$$\sum_{n=1}^{\infty} \frac{\sinh z}{n^2 + 1}$$

is uniformly convergent on the set $\{z : |\mathrm{Re}\, z| \leq 3\}$ *[5]*

Question 5

Let q be the velocity function

$$q(z) = \overline{z} - i.$$

(a) Find the largest region on which q is the velocity function for an ideal flow. *[1]*

(b) Determine the stagnation points of the flow, if there are any. *[1]*

(c) Determine a stream function for the flow, and hence find equations for the streamlines through the points 1 and $-1 - 2i$. *[4]*

(d) Sketch the two streamlines found in part (c), indicating the direction of flow. *[4]*

Question 6

(a) Let $f(z) = z^3 + i$.

(i) Prove that i is a periodic point of f, and determine whether it is attracting (or super-attracting), repelling or indifferent. *[4]*

(ii) Find another periodic point of f *[2]*

(b) Determine whether or not each of the following points c lies in the Mandelbrot set.

(i) $c = -\frac{3}{2}$ *[2]*

(ii) $c = -\frac{3}{2} + \frac{3}{2}i$ *[2]*

Part 2

You should **submit answers to two questions** from Part 2. If you submit answers to all three Part 2 questions, then only Questions 7 and 8 will be marked.

Each question is worth 20%.

Question 7

(a) Let

$$A = \{z : 1 \leq |z| \leq 2\} \quad \text{and} \quad B = \{z : 0 < \operatorname{Arg} z < \pi/2\}.$$

(i) State whether or not each of the sets A, B and $B - A$ is a region. Justify your answers briefly. *[3]*

(ii) State whether or not each of the sets A, B and $B - A$ is compact. Justify your answers briefly. *[3]*

(iii) Prove that the function

$$f(z) = \frac{1}{z}$$

is bounded on A but not bounded on B. *[4]*

(b) Let f be the function

$$f(z) = \overline{z}(1 - z).$$

(i) Prove that

$$f(x + iy) = (x - x^2 - y^2) - iy. \qquad [1]$$

(ii) Use the Cauchy–Riemann Theorem and its converse to show that f is differentiable at 1, but not analytic at 1. *[8]*

(iii) Evaluate $f'(1)$. *[1]*

Question 8

(a) Let $f(z) = \cosh(\sinh z)$.

(i) Find the Taylor series about 0 for f up to the terms in z^4. *[4]*

(ii) Determine the disc of convergence of this Taylor series. *[2]*

(iii) Use part (a)(i) to evaluate the integral

$$\int_C z^3 f(1/z)\, dz,$$

where C is the circle $\{z : |z| = 1\}$. *[4]*

(b) Find the Laurent series about 0 for the function

$$g(z) = \frac{1}{z^2+1}$$

on the region $\{z : |z| > 1\}$, giving four consecutive non-zero terms. *[5]*

(c) (i) Given a positive integer N, prove that there exists a non-constant entire function f that satisfies

$$f(e^{i/n}) = 0,$$

for $n = 1, 2, \ldots, N$. *[2]*

(ii) Prove that there does *not* exist a non-constant entire function g that satisfies

$$g(e^{i/n}) = 0,$$

for *all* positive integers $n = 1, 2, \ldots$. *[3]*

Question 9

(a) Determine

$$\max\{|\exp(z^3)| : |z| \le 3\},$$

and find all points at which the maximum is attained, giving your answers in polar form. *[10]*

(b) Let

$$\mathcal{R} = \{z : |z| < 2\} \quad \text{and} \quad \mathcal{S} = \{z : \operatorname{Re} z < \operatorname{Im} z + 1\}.$$

(i) Sketch the regions $\mathcal{R}$ and $\mathcal{S}$. *[2]*

(ii) Determine a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$. *[6]*

(iii) Using your answer to part (b)(ii), or otherwise, find infinitely many one-to-one conformal mappings from $\mathcal{R}$ onto $\mathcal{S}$. *[2]*

[END OF QUESTION PAPER]